ON THE UNITARIZATION OF HIGHEST WEIGHT REPRESENTATIONS FOR AFFINE KAC-MOODY ALGEBRAS

JUAN GARCIA ESCUDERO and MIGUEL LORENTE
Departamento de Física,Universidad de Oviedo
33007,Oviedo,Spain

ABSTRACT. In a recent paper ([1],[2]) we have classified explicitely all the unitary highest weight representations of non compact real forms of semisimple Lie Algebras on Hermitian symmetric space. These results are necessary in order to construct all the unitary highest weight representations of affine Kac-Moody Algebras following some theorems proved by Jakobsen and Kac ([3],[4]).

1. The Affine Kac-Moody Algebra

Let $\dot{g}$ be a finite dimensional semisimple complex Lie algebra with Chevalley basis $\{H_\alpha, E_\alpha, F_\alpha\}$ with α belonging to the set of simple roots. The elements of the so-called Cartan matrix A are defined by

$$A_{jk} = \alpha_k\left(H_{\alpha j}\right) = \frac{2\left(\alpha_k, \alpha_j\right)}{\left(\alpha_j, \alpha_j\right)}, \qquad j, k = 1, 2, \ldots, \ell$$

where, $(\alpha_j, \alpha_k) = B\left(H_{\alpha_j}, H_{\alpha_k}\right)$, $B(\ ,\ )$ being the Killing form of $\dot{g}$ and H_α an element of the Cartan subalgebra h which is assigned uniquely to each root $\alpha \in \Delta$ by the requirement that $B\left(H_\alpha, H\right) = \alpha(H)$ for all $H \in h$.

The elements in the Chevalley basis satisfy

$$\begin{aligned} \left[H_{\alpha_j}, E_{\alpha_k}\right] &= A_{jk} E_{\alpha_k}\ ,\ \left[H_{\alpha_j}, F_{\alpha_k}\right] = -A_{jk} F_{\alpha_k} \\ \left[E_{\alpha_j}, F_{\alpha_k}\right] &= \delta_{jk} H_{\alpha_k}\ ,\quad \text{for all} \quad \alpha_j, \alpha_k \in \Delta \end{aligned}$$

Every finite dimensional semisimple complex Lie algebra can be constructed from its Cartan matrix A which satisfies the following propierties.

a) $A_{ii} = 2 \quad \forall i, \quad i = 1, \ldots, \ell$
b) $A_{ij} = 0, -1, -2, \quad \text{or } -3 \text{ if } i \neq j, \quad i, j = 1, \ldots, \ell$
c) $A_{ij} = 0$ if and only if $A_{ji} = 0$
d) det A and all proper principal minors of A are positive.

The starting point on the construction of infinite dimensional Kac-Moody algebras is the definition of a generalized Cartan-matrix with elements A_{ij} $(i, j \in I,\ I = 0, 1, \ldots, \ell)$ satisfying

a) $A_{ii} = 2 \qquad \forall i \in I$
b) A_{ij} is either zero or a negative integer for $i \neq j$.
c) $A_{ij} = 0$ if and ony if $A_{ji} = 0$.

A Lie algebra whose Cartan matrix is a generalized Cartan matrix is called a Kac-Moody algebra ([5],[6]). A Kac-Moody algebra is affine if its generalized Cartan matrix is such that det $A = 0$ and all the proper principal minors of A are positive. In the following we restrict ourselves to the affine case.

For a Kac-Moody algebra the Cartan subalgebra h is divided into two parts $h = h' \oplus h''$.

The basis elements of h' are H_{a_j} $(j \in I)$ and h'' is the one-dimensional complementary subspace of h' in h generated by one element that we call d.

The center C in the affine case is one-dimensional. Every element of C is a multiple of h_δ where δ is defined by the following conditions

$$\begin{aligned} \delta(h) &= 0 \qquad \text{for } H \in h' \\ \delta(d) &= 1 \qquad \text{for } d \in h'' \end{aligned}$$

The Cartan subalgebra h has dimension $\ell + 2$. In order to obtain a basis for h^* we need $\ell + 2$ linear functionals. We can take as basis α_k $(k = 0, 1, \ldots, \ell)$, and the linear functional Λ_0 defined as:

$$\begin{aligned} (\Lambda_0, \alpha_k) &= \begin{cases} \frac{1}{2}(\alpha_0, \alpha_0) & \text{if } k = 0 \\ 0 & \text{if } k = 1, 2, \ldots, \ell \end{cases} \\ (\Lambda_0, \delta) &= \tfrac{1}{2}(\alpha_0, \alpha_0) \end{aligned}$$

There exists two types of affine complex Kac-Moody algebras: Untwisted and Twisted. The untwisted ones $g^{(1)}$ may be constructed starting from any simple complex Lie algebra. The twisted affine Kac-Moody algebras $g^{(q)}$ $(q = 2, 3)$ can all be constructed as subalgebras of certain of these untwisted algebras.

i) Untwisted affine Kac-Moody algebras

Let $\dot{g}$ be a simple complex Lie algebra of rank ℓ. A realization of the complex untwisted affine Kac-Moody algebra $g^{(1)}$ is given by

$$g^{(1)} = \mathbb{C}c \oplus \mathbb{C}d \oplus \sum_{j \in Z} z^j \otimes \dot{g}$$

with the following conmutation relations

$$\begin{aligned} &\left[z^j \otimes a, z^k \otimes b\right] = z^{j+k} \otimes [a, b] + j\delta_{j,-k} B(a, b)\, c \quad \forall a, b \in \dot{g} \\ &\left[z^j \otimes a, c\right] = 0 \\ &\left[d, z^j \otimes a\right] = j z^j \otimes a \\ &[d, c] = 0 \end{aligned}$$

ii) Twisted Affine Kac-Moody algebras

Let $\dot{g}$ be a simple complex Lie algebra and τ a rotation of the set of roots of $\dot{g}$. If the rotation τ is not an element of the Weyl group of $\dot{g}$ then there exist an associated outer automorphism Ψ_τ such that $\Psi_\tau(h_\alpha) = h_{\tau(\alpha)}$. We have $\tau^q = -1$ and also

$(\Psi_\tau)^q = 1$ with $q = 2, 3$. The eigenvalues of Ψ_τ are $e^{2\pi ip/q}, \quad p = 0, 1, \ldots, q-1$. Let $\dot{g}_p^{(q)}$ be the eigenspace corresponding to the eigenvalue $e^{2\pi ip/q}$. The Twisted Affine Kac-Moody algebra is then:

$$g^{(q)} = (\not{C}c) \oplus (\not{C}d) \oplus \sum_{p=0}^{q-1} \sum_{\substack{j \\ j\bmod q = p}} \left(z^j \otimes \dot{g}_p^{(q)} \right)$$

2. Highest Weight Representations. The Contravariant Hermitian Form

A subset Δ_+ of Δ is called a set of positive roots if the following propierties are satisfied

i) If $\alpha, \beta \in \Delta_+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta_+$
ii) If $\alpha \in \Delta$ then either α or -α belongs to Δ_+
iii) If $\alpha \in \Delta_+$ then $-\alpha \notin \Delta_+$

For each set of positive roots we may construct a Borel subalgebra $b = \bigoplus_{\alpha\in\Delta_+\cup\{0\}} g_\alpha$. A subalgebra $p \subset g$ such that $b \subset p$ is called a parabolic subalgebra.

Let $U(g)$ denote the universal enveloping algebra of g and let ω be an antilinear anti-involution of g $\left(\text{i.e. } \omega\,[x, y] = [\omega y, \omega x] \quad \text{and } \omega\,(\lambda x) = \overline{\lambda}\omega\,(x)\right)$ such that

$$g = p + \omega p$$

An antilinear anti-involution ω of g is called consistent if $\forall \alpha \in \Delta,\ \omega g_\alpha = g_{-\alpha}$. Now let $e_i,\ f_i,\ h_i \quad (i = 0, \ldots, l)$ be defined as ([5])

$$\begin{array}{lll} e_0 = z \otimes F_{\gamma_r} & f_0 = z^{-1} \otimes E_{\gamma_r} & h_0 = \frac{2}{(\gamma_r, \gamma_r)} c - 1 \otimes H_{\gamma_r} \\ e_i = 1 \otimes E_i & f_i = 1 \otimes F_i & h_i = 1 \otimes H_i \quad i = 1, \ldots, \ell \end{array}$$

where γ_r is the highest positive root.

When $\omega e_i = f_i$ and $\omega h_i = h_i \quad (i = 0, \ldots, l)$ then ω is called the compact antilinear anti-involution and is denoted by w_c.

Let now $\Lambda : p \to \not{C}$ be a 1-dimensional representation of p. A representation $\Pi : g \to g\ell\,(V)$ is called a highest weight representation with highest weight Λ if there exists a vector $\vartheta_\Lambda \in V$ satisfying

a) $\Pi\,(u\,(g))\,\vartheta_\Lambda = V$
b) $\Pi\,(x)\,\vartheta_\Lambda = \Lambda\,(x)\,\vartheta_\Lambda \forall x \in p$

An Hermitian form H on V such that

$$\begin{aligned} &H\,(\vartheta_\Lambda, \vartheta_\Lambda) = 1 \\ &H\,(\Pi\,(x)\,u, v) = H\,(u, \Pi\,(\omega x)\,v) \quad \forall x \in g; \quad u, v \in V \end{aligned}$$

is called contravariant. When H is positive semi-definite, Π is said to be unitarizable.

In the following we construct the Hermitian form H ([3]). We choose a subspace $n \subset g$ such that $g = p \oplus n$. Then we have $U(g) = nU(g) \oplus U(p)$. Let β be the proyection on the second sumand. Let Λ be a 1-dimensional representation of a parabolic subalgebra p (in particular a Borel subalgebra b as in the integrable representations case) satisfying $\Lambda\,(\beta\,(u)) = \overline{\Lambda\,(\beta\,(\omega u))} \quad \forall u \in U\,(g)$.

Let $p^{\Lambda} = \{x \in p / \Lambda(x) = 0\}$. The space

$$M_{p,\omega}(\Lambda) = U(g) / U(g) p^{\Lambda}$$

defines a representation of g on $M_{p,\omega}(\Lambda)$ via left multiplication that is called a (generalized) Verma module and that is a highest weight representation. In addition it can be shown that there exists a unique contravariant hermitian form defined by

$$H(u,v) = \Lambda(\beta(\omega(v))\, u) \text{ for } u, v \in U(g)$$

which is independent of the choice of p.

Let $I(\Lambda)$ denote the Kernel of H on $M_{p,\omega}(\Lambda)$. Then H is nondegenerate on the highest weight module

$$L_{p,\omega}(\Lambda) = M(\Lambda) / I(\Lambda)$$

In the following we will give for each of the unitarizable representations (integrable, elementary and exceptional) the choice of p and ω for which the hermitian form is nondegenerate and positive definite.

3. Integrable representations

Let $\Pi^{st} = \{\alpha_0, \ldots \alpha_\ell\}$ be the standard set of simple roots. The standard set of positive roots ([4]) is $\Delta_+^{st} = \left\{\sum k_i \alpha_i / k_i = 0, 1, 2, \ldots \,;\ \alpha_i \in \Pi^{st}\right\}$ and the corresponding Borel subalgebra is denoted by b^{st}:

$$\begin{aligned} b^{st} &= \mathbb{C} c \oplus \left(1 \otimes \dot{b}\right) \oplus (z \otimes \dot{g}) \oplus \left(z^2 \otimes \dot{g}\right) \oplus \ldots \\ &= \text{ span } \left\{z^k \otimes h_i / k \geq 0, i = 0, \ldots \ell\right\} \oplus \text{span } \left\{z^k \otimes e_i / k \geq 0, i = 0, \ldots \ell\right\} \\ &\oplus \text{ span } \left\{z^k \otimes f_i / k > 0, i = 0, \ldots \ell\right\} \end{aligned}$$

Let $\omega = \omega_c$ and let $\Lambda : b^{st} \to \mathbb{C}$ be a 1-dimensional representation of b^{st} defined by

$$\Lambda(e_i) = 0 \quad \Lambda(h_i) = m_i \in Z_+ \quad (i = 0, \ldots, \ell)$$

These representations are called the integrable highest weight representations. In particular if g is finite-dimensional, these are the finite dimensional representations. The fundamental weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_l$ are such that

$$\left.\begin{array}{lll} \Lambda_j(H_k) &= \frac{2(\Lambda_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk} \\ \Lambda_j(d) &= 0 \end{array}\right\} \quad j, k = 0, \ldots, \ell$$

In this way given the fundamental weights of a finite dimensional Lie algebra $\dot{g}$

$$\dot{\Lambda}_j(H_k) = \frac{2\left(\dot{\Lambda}_j, \alpha_k\right)}{(\alpha_k, \alpha_k)} = \delta_{jk} \quad j, k = 1, \ldots, \ell$$

we can construct the fundamental weights of the Kac-Moody algebra g as extensions of the fundamental weights $\dot{\Lambda}$ in the following way:

$$\begin{aligned} \Lambda_j &= \dot{\Lambda}_j + \mu_j \Lambda_0 \\ \mu_j &= -\sum_{k=1}^{\ell} A_{ok} \left((\dot{A})^{-1} \right)_{kj} \end{aligned}$$

$\dot{A}$ being the Cartan matrix of $\dot{g}$ and Λ_0 the linear functional defined in paragraph 1.

Every integrable highest weight representation is unitarizable.

4. Elementary representations

We know that for a finite dimensional simple Lie algebra $\dot{g}$ an infinite dimensional highest weight representation is unitarizable only if $\dot{\omega}$ is a consistent antilinear anti-involution corresponding to a hermitian symmetric space (see [7],[8]).

The remaining unitarizable representations can be constructed only for Kac-Moody algebras related to these type of finite dimensional Lie algebras.

Let $\dot{b} = \bigoplus_{\alpha \in \dot{\Delta}_+ \cup \{0\}} \dot{g}_\alpha$ be a Borel subalgebra of the finite dimensional Lie algebra $\dot{g}$.

Consider the parabolic subalgebra (called "natural")

$$p^{\text{nat}} = z^n \otimes \dot{b} = \text{span}\left\{ z^n \otimes h_i, z^n \otimes e_i \right\}, \quad n \in Z, \quad i = 1, \ldots, \ell$$

Take a Cartan decomposition of the Lie algebra $\dot{g}$ corresponding to a hermitian symmetric space:

$$\dot{g} = \dot{p}^- \oplus \dot{k} \oplus \dot{p}^+, \quad \dot{k} = \dot{k}^- \oplus \dot{\eta} \oplus \dot{k}^+$$

where $\dot{p}$ and $\dot{k}$ are the subspace and the subalgebra corresponding to non-compact and compact roots respectively.

We define an antilinear anti-involution ω of g by

$$\left. \begin{aligned} \omega\left(z^n \otimes k_i^+\right) &= z^{-n} \otimes k_i^- \quad i = 2, \ldots, \ell \\ \omega\left(z^n \otimes p_1^+\right) &= -z^{-n} \otimes p_1^- \\ \omega\left(z^n \otimes h_i\right) &= z^{-n} \otimes h_i \quad i = 0, 1, \ldots, \ell \end{aligned} \right\} \quad n \in Z$$

where $k_2^+, \ldots, k_l^+$ belong to $\dot{k}^+$ and where p_1^+ belongs to the root space corresponding to the unique simple non compact root. In the previous notation $e_1 = p_1^+$ and $e_i = k_i^+$ for $i = 2, \ldots, \ell$.

Consider now a set of highest weights $\Lambda_1, \ldots, \Lambda_N$ corresponding to unitarizable highest weight modules for the hermitian symmetric space $\dot{g}$ (for an explicit calculation see [1],[2]). Define a representation $p^{\text{nat}} \to \mathbb{C}$ by

$$\Lambda\left(z^k \otimes x\right) = \sum_{i=1}^{N} C_i^k \Lambda_i(x)$$

for $x \in \dot{b}$ and $C_k^i \in \mathbb{C}$ with $\left|C_k^i\right| = 1$.

Then the resulting representation is called "elementary" and it is unitarizable.

5. Exceptional representations

Another class of unitary representations (called "exceptional") are constructed in this paragraph for the Kac-Moody algebra $z^k \otimes su(n,1) \quad k \in Z, \quad n \geq 1$.

Let $\dot{g} = su(n,1)$ and let be a Cartan decomposition $\dot{g} = \dot{k} \oplus \dot{p}$. Then $\dot{h} = \dot{k}_1 \cap \dot{h} \oplus R\dot{h}_c$ where $\dot{k}_1 = \left[\dot{k}, \dot{k}\right]$ and $\dot{h}_c$ belongs to the center of $\dot{g}$.

We take a realization of $g = z^k \otimes su(n,1)$ in terms of matrices $(a_{ij}(z)) \quad i,j = 0,\ldots,n$. The matrix elements are of the form $a(z) = \sum\limits_{n\in Z} a_n e^{in\theta}$ with $z = e^{i\theta}$. We will use the notation $\overline{a}(z) = \sum\limits_{n\in Z} \overline{a}_n e^{in\theta}$. Let $p = \{(a_{ij}(z)) \in g/a_{ij} = 0 \quad \text{if } i > j\}$ be the parabolic subalgebra. The antilinear anti-involution acts in this case as

$$\begin{array}{ll} \omega\left(z^k \otimes h_c\right) & : \omega\left(a_{00}(z)\right) = \overline{a}_{00}\left(z^{-1}\right) \\ \omega\left(z^k \otimes \dot{p}^+\right) & : \omega\left(a_{0j}(z)\right) = -\overline{a}_{j0}\left(z^{-1}\right) \quad j = 1,\ldots,n \\ \omega\left(z^k \otimes \dot{k}_1\right) & : \omega\left(a_{ij}(z)\right) = \overline{a}_{ji}\left(z^{-1}\right) \quad i,j = 1,\ldots,n \end{array}$$

Define a representation $\Lambda : p \to \mathbb{C}$ by

$$\begin{array}{rcl} \Lambda\left(a_{ij}(z)\right) & = & 0 \qquad i,j = 1,\ldots,n \\ \Lambda\left(a_{0j}(z)\right) & = & 0 \qquad j = 1,\ldots,n \\ \Lambda\left(a_{00}(z)\right) & = & -\int_{s^1} a_{00}\left(e^{i\theta}\right) d\mu(\theta) = -\varphi\left(a_{00}(z)\right) \end{array}$$

where $\mu(\theta)$ is a positive Radon mesure defined in the unit circle s^1 and infinitely supported.

It can be shown (see [3]) that the hermitian form H (we remind that it is completely determined by giving w, p and a representation of p) is positive definite and then the corresponding representation in the space $L_{p,w}(L)$ is unitary.

6. Tensor products

The only remaining possibility in order to complete the set of all unitarizable representations for affine Kac-Moody algebras is the corresponding to the highest component of a tensor product of an elementary with an exceptional representation for $z^k \otimes su(n,1)$ (see [4]).

Explicit results concerning the construction of these unitary representations will be given in a forthcoming paper.

References

1 García-Escudero, J. and Lorente, M.: 'Highest Weight Unitary Modules for Non-Compact Groups and Applications to Physical Problems', *Symmetries in Science V* (ed. B. Gruber et al.) Plenum Publishing. New York (1991)

2 García-Escudero, J. and Lorente, M.: 'Classification of Unitary Highest Weight Representations for Non-compact Real Forms'. *J. Math. Phys.*, 781 – 790 (1990).

3 Jakobsen, H.P. and Kac, V.: 'A new class of unitarizable highest weight representations of infinite dimensional Lie algebras', in *Non-Linear Equations in Classical and Quantum Field Theory* (ed. N. Sánchez). Lect. Notes in Phys. 226, Springer Verlag (1985).

4 Jakobsen, H.P. and Kac, V. 'A new class of unitarizable highest weight representations of infinite dimensional Lie algebras II'. *J. Funct. Anal.* 82 (1989).

5 Kac, V.G. Infinite dimensional Lie algebras; Cambridge University Press third edition, Cambridge 1990 .

6 Cornwell, J.F. Group Theory in Physics, Vol III. Academic Press (1989).

7 Jakobsen, H.P. 'Hermitian Symmetric Spaces and their Unitary Highest Weight Modules'. *J. Funct. Anal.* $52, 385 - 412$ (1983) .

8 Enright, T; Howe, R.; Wallach, N. 'A classification of Unitary Highest Weight Modules, in Representation Theory of Reductive Groups' (ed. P. Trombi) *Progress in Math.* 40. Birkhaüser, Boston (1983).